

A CLASSIFICATION OF POSTCRITICALLY FINITE NEWTON MAPS

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ABSTRACT. The dynamical classification of rational maps is a central concern of holomorphic dynamics. Much progress has been made, especially on the classification of polynomials and some approachable one-parameter families of rational maps; the goal of finding a classification of general rational maps is so far elusive. Newton maps (rational maps that arise when applying Newton’s method to a polynomial) form a most natural family to be studied from the dynamical perspective. Using Thurston’s characterization and rigidity theorem, a complete combinatorial classification of postcritically finite Newton maps is given in terms of a finite connected graph satisfying certain explicit conditions.

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1. INTRODUCTION

The past three decades have seen tremendous progress in the understanding of holomorphic dynamics. This is largely due to the fact that the complex structure provides enough rigidity, allowing many interesting questions to be reduced to tractable combinatorial problems.

To understand the dynamics of rational maps, an important first step is to understand the dynamics of postcritically finite maps, namely the maps where each critical point has finite forward orbit. Thurston’s “Fundamental Theorem of Complex Dynamics” is available in this setting, providing an important characterization and rigidity theorem for postcritically finite branched covers that arise from rational maps. Also, the postcritically finite maps are the structurally important ones, and conjecturally, the set of maps

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that are quasiconformally equivalent (in a neighborhood of the Julia set) to such maps are dense in parameter spaces [McM94, Conjecture 1.1].

Polynomials form an important and well-understood class of rational functions. In this case, the point at infinity is completely invariant, and is contained in a completely invariant Fatou component. This permits enough dynamical structure so that postcritically finite polynomials may be described in finite terms, e.g. using external angles at critical values or finite Hubbard trees. A complete classification of polynomials has been given [BFH92, Poi93].

Classification results for more general rational functions are rare, and mostly concern the case of one-dimensional families. However, one family that has been studied with some success in the past is the family of Newton maps.

Definition 1.1 (Newton map). *A rational function $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is called a Newton map if there is some complex polynomial $p(z)$ so that $f(z) = z - \frac{p(z)}{p'(z)}$ for all $z \in \mathbb{C}$.*

Denote such a Newton map by N_p . Newton maps of degree 1 and 2 are trivial and thus excluded from our entire discussion.

Note that N_p is precisely the function that is iterated when Newton's method is used to find the roots of the polynomial p . Each root of p is an attracting fixed point of the Newton map, and the point at infinity is a repelling fixed point. The algebraic number of roots of p is the degree of p , while the geometric number of roots of p (ignoring multiplicities) equals the degree of N_p . The space of degree d Newton maps considered up to affine conjugacy has $d - 2$ degrees of freedom, given by the location of the d roots of p after normalization. The space of degree d complex polynomials up to affine conjugacy has $d - 1$ degrees of freedom, given by the $d + 1$ coefficients after normalization. Thus it is clear that Newton maps form a substantial subclass of rational maps. All the finite fixed points of a postcritically finite Newton map are in fact superattracting, and this only occurs when all roots of p are simple.

An important first theorem on Newton maps is the following characterization of Newton maps in terms of fixed point multipliers.

Proposition 1.2 (Head's theorem). [Hea87] *A rational map f of degree $d \geq 3$ is a Newton map if and only if for each fixed point $\xi \in \mathbb{C}$, there is an integer $m \geq 1$ so that $f'(\xi) = (m - 1)/m$.*

This condition on multipliers forces ∞ to be a repelling fixed point by the holomorphic fixed point formula.

There are a number of partial classification theorems for postcritically finite Newton maps. Tan Lei has given a classification of cubic Newton maps in terms of matings and captures (or alternatively in terms of abstract graphs [Tan97]; see also earlier work by Head [Hea87]). Luo produced a similar combinatorial classification for Newton maps of arbitrary degree having a single non-fixed critical value that is either periodic or eventually maps to a fixed critical point [Luo95].

Rückert recently classified the postcritically *fixed* Newton maps for arbitrary degree, namely those Newton maps whose critical points are eventually

fixed [Rüc06] (see also [MRS]). The fundamental piece of combinatorial data is the *channel diagram* Δ which is constructed in [HSS01]. It is a graph in the Riemann sphere whose vertices are given by the fixed points of the Newton map and whose edges are given by all accesses of the immediate basins of roots connecting the roots to ∞ (see the solid lines of Figure 1). To capture the behavior of nonperiodic critical points that eventually map to the channel diagram, it is natural to consider the graph $N_p^{-n}(\Delta)$ for some integer n . However this graph is not necessarily connected (see Figure 1 for example), and so the *Newton graph of level n* associated to N_p is defined to be the component of $N_p^{-n}(\Delta)$ that contains Δ . Rückert proves that for any postcritically finite Newton map N_p , there is some level n so that the Newton graph of level n contains all critical points that eventually map to the channel diagram (this fact is non-trivial because preimage components of the channel diagram were discarded). For minimal n this component is called the *Newton graph* in the context of postcritically fixed maps, and the data consisting of this graph equipped with a graph map inherited from the dynamics of the Newton map is enough to classify postcritically *fixed* Newton maps

Building on the thesis [Mik11], we classify postcritically *finite* Newton maps. In [LMS] a finite graph containing the postcritical set was constructed for a postcritically finite Newton map. The graph is composed of three types of pieces:

- the Newton graph (which contains the channel diagram) is used to capture the behavior of critical points that are eventually fixed.
- Hubbard trees are used to give combinatorial descriptions of renormalizations at periodic non-fixed postcritical points. Preimages of the Hubbard trees are taken to capture the behavior of critical points whose orbits intersect the Hubbard trees.
- Newton rays (single edges comprised of a sequence of infinitely many preimages of channel diagram edges) are used to connect all Hubbard trees and their preimages to the Newton graph.

The construction of these three types of edges is not given here, but an example is provided in Figures 2 and 3.

The restriction of the Newton map to this “extended Newton graph” yields a graph self-map, and the resulting dynamical graphs are axiomatized (as abstract extended Newton graphs; see Definition 4.5).

Theorem 1.3 (Newton maps to graphs). [LMS, Theorem 1.2] *For any extended Newton graph $\Delta_N^* \subset \widehat{\mathbb{C}}$ associated to a postcritically finite Newton map N_p , the pair (Δ_N^*, N_p) satisfies the axioms of an abstract extended Newton graph.*

It must be emphasized that arbitrary choices were made in the construction of the Newton rays, necessitating a rather subtle but natural combinatorial equivalence relation on our way to a classification.

Our first main result is that every abstract extended Newton graph is realized by a unique Newton map (up to affine conjugacy), and is proven using Thurston’s theorem. In the following theorem statement, \bar{f} denotes the unique extension (up to Thurston equivalence) of the graph map f to a

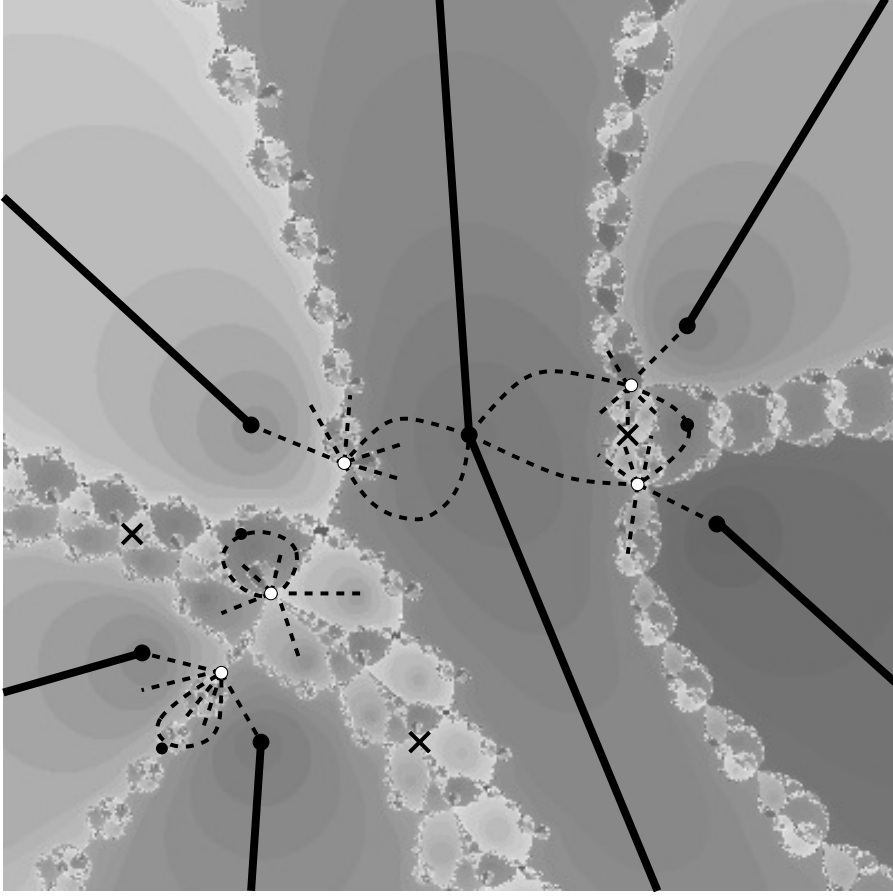


FIGURE 1. The dynamical plane of a degree 6 Newton map. The channel diagram is drawn using thick black lines. The first preimage of the channel diagram consists of the channel diagram together with all edges drawn using dashed lines (all edges are drawn up to homotopy rel endpoints). The white dots represent poles. The three x's represent critical points outside of the channel diagram. It should be observed that the first preimage of the channel diagram is not connected.

branched cover of the whole sphere, and the set of vertices of a graph Γ is denoted Γ' .

Theorem 1.4 (Graphs to Newton maps). *Let (Σ, f) be an abstract extended Newton graph (as in Definition 4.5). Then there is a postcritically finite Newton map N_p , unique up to affine conjugacy, with extended Newton graph Δ_N^* so that the marked branched covers (\bar{f}, Σ') and $(N_p, (\Delta_N^*)')$ are Thurston equivalent.*

Denote by \mathcal{N} the set of postcritically finite Newton maps up to affine conjugacy, and by \mathcal{G} we denote the set of abstract extended Newton graphs under the graph equivalence of Definition 5.7. It follows from the statements of Theorem 1.3 and 1.4 that there are well-defined maps $\mathcal{F} : \mathcal{N} \rightarrow \mathcal{G}$ and

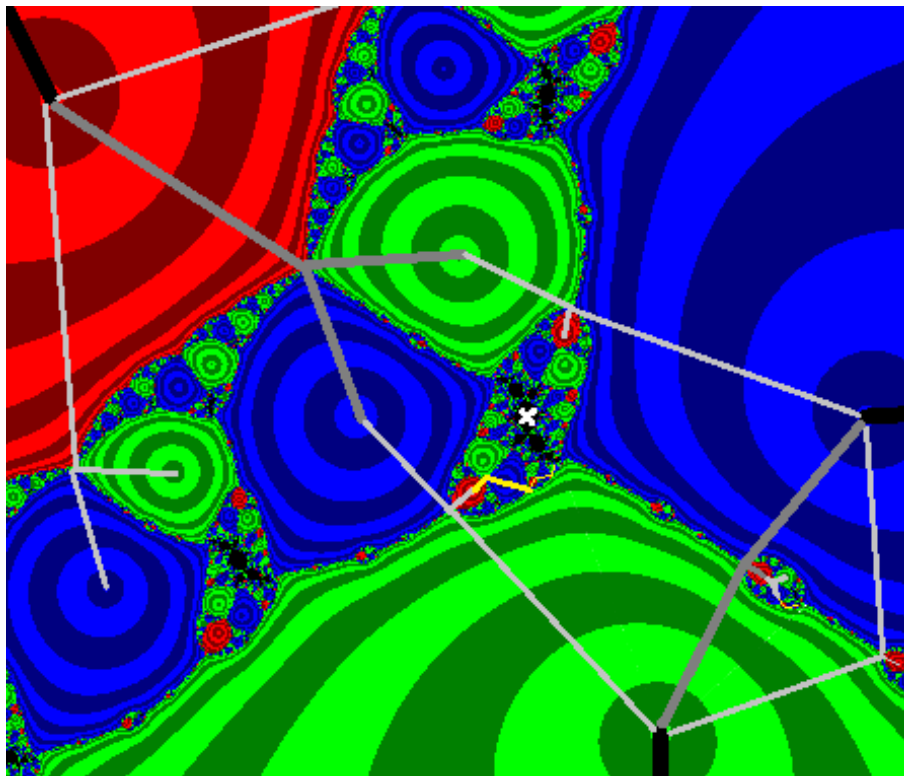


FIGURE 2. The dynamical plane of a cubic Newton map N_p (produced by Mandel) with extended Newton graph overlaid (omitting Hubbard trees for now). The centers of the biggest red, green, and blue basins are fixed critical points. Thick black edges indicate the channel diagram Δ , medium width dark gray edges indicate the part of the Newton graph of level one Δ_1 that is not already in Δ , and thin light gray edges indicate the part of the Newton graph of level two that is not in Δ_1 . The white “x” denotes a free critical point which has period 6 and is contained in a little rabbit K_R . The non-separating fixed point of K_R lies at the endpoint of the $2/3$ ray in the fixed green basin while the same holds for $N_p(K_R)$ with the $1/3$ ray (nether ray depicted). There are two Newton rays drawn in yellow connecting the Newton graph to both of these filled Julia sets (the ray for $N_p(K_R)$ is very small). Note that $N_p^{\circ 2}(K_R) = K_R$. The combinatorial invariant for N_p consists of all depicted edges union the Hubbard trees for K_R and $N_p(K_R)$. See zoom of $N_p(K_R)$ in Figure 3.

$\mathcal{F}' : \mathcal{G} \rightarrow \mathcal{N}$ respectively. It will be shown that the mappings \mathcal{F} and \mathcal{F}' are bijective, and inverses of each other, yielding the following theorem.

Theorem 1.5 (Combinatorial classification). *There is a natural bijection between the set of postcritically finite Newton maps (up to affine conjugacy)*

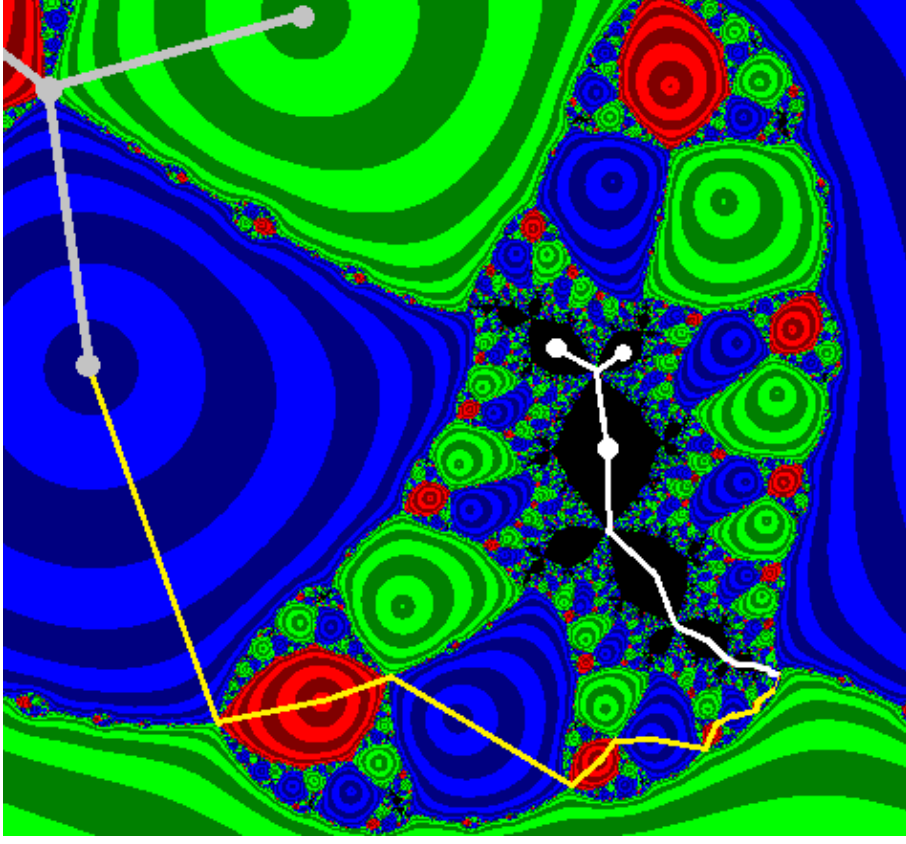


FIGURE 3. Magnification of Figure 2 around $N_p(K_R)$ with Hubbard tree drawn. The gray edges are a portion of the Newton graph in Figure 2. The yellow edge is a period 2 Newton ray. The white edges indicate the Hubbard tree for $N_p(K_R)$. The white points indicate postcritical points. The intersection of the Hubbard tree and the Newton ray consists precisely of the endpoint of the $2/3$ ray in the lower-most green basin (which is in fact the immediate basin of a root).

and the set of abstract extended Newton graphs (up to combinatorial equivalence) so that for every abstract extended Newton graph (Σ, f) , the associated postcritically finite Newton map has the property that any associated extended Newton graph is equivalent to (Σ, f) .

Remark 1.6. This paper not only provides a classification of the largest non-polynomial family of rational maps so far, it also lays foundations for classification results in a substantially larger context (current work in progress): the fundamental property of the dynamics that we are using is that Fatou components have a common accessible boundary point at infinity, as well as the preimages of these Fatou components. A basic ingredient in more general classification results builds on periodic Fatou components with common accessible boundary points, and for these our methods will be a key ingredient. We would also like to mention current work by Mamayusupov [Mam15] that

classifies those rational maps that arise as Newton maps of transcendental entire functions; this classification builds upon our result.

We conclude this introduction with a remark illustrating our point of view that Newton maps are a “pioneering class of rational functions”, the first large one after polynomials, that might guide the way towards understanding more general rational maps. The well known “Fatou-Shishikura-inequality” says that any rational map has no more non-repelling periodic orbits than it has critical points. However, there is a more refined version for polynomials: every non-repelling periodic orbit has its own critical point; this is a more precise “local” version that obviously implies the global inequality $[BCL^+]$. It follows from our methods that this inequality also holds for Newton maps. Roughly speaking, the idea is that non-repelling fixed points are roots and thus attracting fixed points, for which the claim is trivially true, while non-repelling cycles of higher period must be associated to domains of renormalization and hence to polynomial-like maps (in analogy to our results in Section 4.3 of [LMS] which are only worked out in detail for postcritically finite maps), and the result thus follows from the corresponding result on polynomials, as in $[BCL^+]$.

Structure of this paper: Section 2 introduces Thurston’s characterization and rigidity theorem for postcritically finite branched covers. This theorem asserts that a topological branched cover that has no obstructing multicurves is uniquely realized by a rational map (under a mild assumption that is irrelevant for our purposes). Since it is often very hard to show directly that a cover is unobstructed—for this purpose, a useful theorem of Pilgrim and Tan controlling the location of obstructions will be presented. Section 3 presents a result on how to extend certain kinds of graph maps to branched covers on the whole sphere.

Section 4 defines the abstract extended Newton graph, which will be shown to be a complete invariant for postcritically finite Newton maps. The equivalence on such graphs is defined in Section 5, and the connection between this combinatorial equivalence and Thurston equivalence is described.

Section 6 proves Theorem 1.4 by showing that abstract extended Newton graphs equipped with their graph self-maps extend to branched covers of the sphere that are unobstructed.

Section 7 proves Theorem 1.5, establishing the combinatorial classification of postcritically finite Newton maps.

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2. THURSTON THEORY ON BRANCHED COVERS

We will be using Thurston’s theorem to prove that the combinatorial model for postcritically finite Newton maps is realized by a rational map, and we present the requisite background in this section. As one observes from the statement of Thurston’s theorem below, this amounts to showing that the combinatorial model has no obstructing multicurves. There are infinitely many multicurves in a sphere with four or more marked points, so a priori it is very hard to show obstructions do not exist. However, the

“arcs intersecting obstructions” theorem of Pilgrim and Tan can in some cases drastically reduce the possible locations of obstructions.

Let $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be an orientation preserving branched cover from the two-sphere to itself. Denote the set of critical points of f by C_f . Define the postcritical set P_f as follows:

$$P_f := \bigcup_{n \geq 1} f^{\circ n}(C_f).$$

The map f is said to be *postcritically finite* if the set P_f is finite.

A *marked branched cover* is a pair (f, X) , where $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is an orientation preserving branched cover and X is a finite set containing P_f such that $f(X) \subset X$.

Definition 2.1 (Thurston equivalence of marked branched covers). *Two marked branched covers (f, X) and (g, Y) are Thurston equivalent if there are two orientation preserving homeomorphisms $\phi_1, \phi_2 : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that*

$$\phi_1 \circ f = g \circ \phi_2$$

and there exists an homotopy $\Phi : [0, 1] \times \mathbb{S}^2 \rightarrow \mathbb{S}^2$ with $\Phi(0, \cdot) = \phi_1$ and $\Phi(1, \cdot) = \phi_2$ such that $\Phi(t, \cdot)|_X$ is constant in $t \in [0, 1]$ with $\Phi(t, X) = Y$. If ϕ_1 and ϕ_2 are homotopic to the identity map, the marked branched covers are said to be homotopic.

We say that a simple closed curve γ is a *simple closed curve* in (\mathbb{S}^2, X) if $\gamma \subset \mathbb{S}^2 \setminus X$. Such a γ is *essential* if both components of the complement $\mathbb{S}^2 \setminus \gamma$ contain at least two points of X . Let γ_0, γ_1 be two simple closed curves in (\mathbb{S}^2, X) . We say that γ_0 and γ_1 are *isotopic relative to X* , written $\gamma_0 \simeq_X \gamma_1$, if there exists a continuous, one-parameter family of simple closed curves in (\mathbb{S}^2, X) joining γ_0 and γ_1 . We use $[\gamma]$ to denote the isotopy class of a simple closed curve γ . A *multicurve* is a collection of pairwise disjoint and non-isotopic essential simple closed curves in (\mathbb{S}^2, X) . A multicurve Π is said to be *f -stable* if for every $\gamma \in \Pi$, every essential connected component of $f^{-1}(\gamma)$ is isotopic relative to X to some element of Π .

Definition 2.2 (Thurston linear map). *For every f -stable multicurve Π we define the corresponding Thurston linear transform $f_\Pi : \mathbb{R}^\Pi \rightarrow \mathbb{R}^\Pi$ as follows:*

$$f_\Pi(\gamma) = \sum_{\gamma' \subset f^{-1}(\gamma)} \frac{1}{\deg(f|_{\gamma'} : \gamma' \rightarrow \gamma)} [\gamma'],$$

where $[\gamma']$ denotes the element of Π isotopic to γ' , if it exists. If there are no such elements, the sum is taken to be zero. Denote by λ_Π the largest eigenvalue of f_Π (known to exist by the Perron-Frobenius theorem).

Suppose that Π is a stable multicurve. A multicurve Π is called a *Thurston obstruction* if $\lambda_\Pi \geq 1$. A real-valued $n \times n$ matrix A is called *irreducible* if for every entry (i, j) , there exists an integer $k > 0$ such that $A_{i,j}^k > 0$. A multicurve Π is said to be *irreducible* if the matrix representing the linear transform f_Π is irreducible.

Our statement of Thurston’s theorem uses the notion of a hyperbolic orbifold. We omit the definition, referring the reader to [DH93] while observing

that there are few cases where O_f is not hyperbolic, and that O_f is always hyperbolic if f has at least three fixed branched points. The latter is always the case for Newton maps of degree $d \geq 3$, so the restriction to hyperbolic orbifolds is of no concern to us.

Theorem 2.3 (Thurston's theorem [DH93, BCT14]). *A marked branched cover (f, X) with hyperbolic orbifold is Thurston equivalent to a marked rational map if and only if (f, X) has no Thurston obstruction. Furthermore, if (f, X) is unobstructed, the marked rational map is unique up to Möbius conjugacy.*

We now present a theorem of Pilgrim and Tan [PT98] that will be used in Section 6 to show that certain marked branched covers arising from graph maps do not have obstructions and are therefore equivalent to rational maps by Thurston's theorem. First some notation will be introduced.

Assume that (f, X) is a marked branched cover of degree $d \geq 3$. An *arc* in (\mathbb{S}^2, X) is a continuous map $\alpha : [0, 1] \rightarrow \mathbb{S}^2$ such that $\alpha(0)$ and $\alpha(1)$ are in X , the map α is injective on $(0, 1)$, and $X \cap \alpha((0, 1)) = \emptyset$. A set of pairwise non-isotopic arcs in (\mathbb{S}^2, X) is called an *arc system*. The following intersection number is used in the statement of Theorem 2.5; we use the symbol \simeq to denote isotopy relative to X .

Definition 2.4 (Intersection number). *Let α and β each be an arc or a simple closed curve in (\mathbb{S}^2, X) . Their intersection number is*

$$\alpha \cdot \beta := \min_{\alpha' \simeq \alpha, \beta' \simeq \beta} \# \{(\alpha' \cap \beta') \setminus X\}.$$

The intersection number extends bilinearly to arc systems and multicurves.

For an arc system Λ , we introduce a linear map $f_\Lambda : \mathbb{R}^\Lambda \rightarrow \mathbb{R}^\Lambda$, which is an unweighted analogue of the Thurston linear map for multicurves. For $\lambda \in \Lambda$, let

$$f_\Lambda(\lambda) := \sum_{\lambda' \subset f^{-1}(\lambda)} [\lambda']_\Lambda,$$

where $[\lambda']_\Lambda$ denotes the element of Λ homotopic to λ' rel X (the sum is taken to be zero if there are no such elements). It is said that Λ is *irreducible* if the matrix representing f_Λ is irreducible.

Denote by $\tilde{\Lambda}(f^{\circ n})$ the union of those components of $f^{-n}(\Lambda)$ that are isotopic to elements of Λ relative X , and define $\tilde{\Pi}(f^{\circ n})$ for a multicurve Π analogously. The following theorem (Theorem 3.2 of [PT98]) gives control on the location of irreducible Thurston obstructions by asserting that they may not intersect certain preimages of irreducible arc systems.

Theorem 2.5 (Arcs intersecting obstructions). [PT98] *Let (f, X) be a marked branched cover, Π an irreducible Thurston obstruction, and Λ an irreducible arc system. Suppose furthermore that $\#(\Pi \cap \Lambda) = \Pi \cdot \Lambda$. Then exactly one of the following is true:*

- (1) $\Pi \cdot \Lambda = 0$ and $\Pi \cdot f^{-n}(\Lambda) = 0$ for all $n \geq 1$.
- (2) $\Pi \cdot \Lambda \neq 0$ and for $n \geq 1$, each component of Π is isotopic to a unique component of $\tilde{\Pi}(f^{\circ n})$. The mapping $f^{\circ n} : \tilde{\Pi}(f^{\circ n}) \rightarrow \Pi$ is a homeomorphism and $\tilde{\Pi}(f^{\circ n}) \cap (f^{-n}(\Lambda) \setminus \tilde{\Lambda}(f^{\circ n})) = \emptyset$. The same is true when interchanging the roles of Π and Λ .

3. EXTENDING MAPS ON FINITE GRAPHS

All graphs in this paper are embedded in \mathbb{S}^2 . The set of vertices of a graph Γ is denoted Γ' . A sufficient condition under which a finite graph map has a unique extension to a map on the whole sphere (up to equivalence) is presented following [BFH92, Chapter 5]. The Alexander Trick is foundational to such results.

Lemma 3.1 (Alexander trick). *Let $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be an orientation-preserving homeomorphism. Then there exists an orientation preserving homeomorphism $\bar{h} : \mathbb{D} \rightarrow \mathbb{D}$ such that $\bar{h}|_{\mathbb{S}^1} = h$. The map \bar{h} is unique up to isotopy relative \mathbb{S}^1 .*

Let Γ_1, Γ_2 be connected finite embedded graphs. A continuous map $f : \Gamma_1 \rightarrow \Gamma_2$ is called a *graph map* if it is injective on each edge of the graph Γ_1 and the forward and backward images of vertices are vertices.

Definition 3.2 (Regular extension). *Let $f : \Gamma_1 \rightarrow \Gamma_2$ be a graph map. An orientation-preserving branched cover $\bar{f} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is called a regular extension of f if $\bar{f}|_{\Gamma_1} = f$ and \bar{f} is injective on each component of $\mathbb{S}^2 \setminus \Gamma_1$.*

It follows immediately that \bar{f} may only have critical points at the vertices of Γ_1 . The following lemma gives a condition under which two extensions are isotopic relative to the vertices.

Lemma 3.3 (Unique extendability). [BFH92, Corollary 6.3] *Let $f, g : \Gamma_1 \rightarrow \Gamma_2$ be two graph maps that coincide on the vertices of Γ_1 and for each edge e in Γ_1 we have $f(e) = g(e)$ as a set. Suppose that f and g have regular extensions $\bar{f}, \bar{g} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$. Then there exists a homeomorphism $\psi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$, isotopic to the identity relative the vertices of Γ_1 , such that $\bar{f} = \bar{g} \circ \psi$.*

We must establish some notation for the following proposition. Let $f : \Gamma_1 \rightarrow \Gamma_2$ be a graph map. For each vertex v of Γ_1 choose a neighborhood $U_v \subset \mathbb{S}^2$ containing v such that all edges of Γ_1 intersect ∂U_v no more than once; we may assume without loss of generality that in local coordinates, U_v is a round disk of radius 1 that is centered at v , that the intersection of any edge with U_v is either empty or a radial line segment, and that $f|_{U_v}$ is length-preserving. Make analogous assumptions for Γ_2 .

We describe how to explicitly extend f to each U_v . For a vertex $v \in \Gamma_1$, let γ_1 and γ_2 be two adjacent edges ending there. In local coordinates, these are radial lines at angles Θ_1, Θ_2 where $0 < \Theta_2 - \Theta_1 \leq 2\pi$ (if v is an endpoint of Γ_1 , then set $\Theta_1 = 0, \Theta_2 = 2\pi$). In the same way, choose arguments Θ'_1, Θ'_2 for the image edges in $U_{f(v)}$ and extend f to a map \tilde{f} on $\Gamma_1 \cup \bigcup_v U_v$ by setting

$$(\rho, \Theta) \mapsto \left(\rho, \frac{\Theta'_2 - \Theta'_1}{\Theta_2 - \Theta_1} \cdot \Theta \right),$$

where (ρ, Θ) are polar coordinates in the sector bounded by the rays at angles Θ_1 and Θ_2 . In particular, sectors are mapped onto sectors in an orientation-preserving way.

A characterization of graph maps with regular extensions is now given in terms of the local extension just constructed about vertices.

Proposition 3.4. [BFH92, Proposition 6.4] *A graph map $f : \Gamma_1 \rightarrow \Gamma_2$ has a regular extension if and only if for every vertex $y \in \Gamma_2$ and every component U of $\mathbb{S}^2 \setminus \Gamma_1$, the extension \tilde{f} is injective on*

$$\bigcup_{v \in f^{-1}(y)} U_v \cap U.$$

The fundamental combinatorial object in our classification of Newton maps is a finite graph Σ equipped with a self-map $f : \Sigma \rightarrow \Sigma$ (Definition 4.5). Strictly speaking, f is in general not a graph map since Newton ray edges contain finitely many preimages of vertices in the Newton graph that are not vertices in Σ (these vertices are purposely ignored on our way to producing a finite graph). This motivates the following weaker definition where the condition on preimages of vertices has been dropped.

Definition 3.5 (Weak graph map). *A continuous map $f : \Gamma_1 \rightarrow \Gamma_2$ is called a weak graph map if it is injective on each edge of the graph Γ_1 and the forward image of vertices are vertices.*

Remark 3.6. Given a weak graph map $f : \Gamma_1 \rightarrow \Gamma_2$, the combinatorics of the domain can be slightly altered to produce a graph map $\hat{f} : \hat{\Gamma}_1 \rightarrow \Gamma_2$ in the following natural way. We take the graph $\hat{\Gamma}_1$ to have vertices given by $f^{-1}(\Gamma'_2)$, and edges given by the closures of complementary components of $\Gamma_1 \setminus f^{-1}(\Gamma'_2)$. We simply take $\hat{f} = f$.

4. ABSTRACT EXTENDED NEWTON GRAPH

In [LMS, Section 6.1], we extracted from every postcritically finite Newton graph an extended Newton graph, and we axiomatized these graphs in [LMS, Section 7]. In this section we review the definition of the abstract extended Newton graph which will be used in Section 7 of the present work to classify postcritically finite Newton maps. Abstract extended Newton graphs consist of three pieces: abstract Newton graphs, abstract extended Hubbard trees, and abstract Newton rays connecting the first two objects.

The definition of abstract extended Hubbard tree was given in Definition 4.4 of [LMS], and will not be repeated here. We simply note that it is the usual definition of degree d abstract Hubbard tree from [Poi93], where the set of marked points includes all periodic points of periods up to some fixed length n (since postcritically finite Newton maps cannot have parabolic cycles, the number of periodic points of period i equals d^i). Such an abstract extended Hubbard tree is said to have *cycle type n* .

To define the abstract Newton graph, it is necessary to first define the abstract channel diagram.

Definition 4.1. *An abstract channel diagram of degree $d \geq 3$ is a graph $\Delta \subset \mathbb{S}^2$ with vertices $v_\infty, v_1, \dots, v_d$ and edges e_1, \dots, e_l that satisfies the following:*

- $l \leq 2d - 2$;
- each edge joins v_∞ to some v_i for $i \in \{1, 2, \dots, d\}$;
- each v_i is connected to v_∞ by at least one edge;
- if e_i and e_j both join v_∞ to v_k , then each connected component of $\mathbb{S}^2 \setminus \overline{e_i \cup e_j}$ contains at least one vertex of Δ .

Definition 4.2 (Abstract Newton graph). *Let $\Gamma \subset \mathbb{S}^2$ be a connected finite graph, Γ' the set of its vertices and $f : \Gamma \rightarrow \Gamma$ a graph map. The pair (Γ, f) is called an abstract Newton graph of level N_Γ if it satisfies the following:*

- (1) *There exists $d_\Gamma \geq 3$ and an abstract channel diagram $\Delta \subsetneq \Gamma$ of degree d_Γ such that f fixes each vertex and each edge (pointwise) of Δ .*
- (2) *There is an extension of the graph map f to a branched cover $\tilde{f} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that the following conditions (3) – (6) are satisfied.*
- (3) *Γ equals the component of $\tilde{f}^{-N_\Gamma}(\Delta)$ that contains Δ .*
- (4) *If $v_\infty, v_1, \dots, v_{d_\Gamma}$ are the vertices of Δ , then $v_i \in \overline{\Gamma \setminus \Delta}$ if and only if $i \neq \infty$. Moreover, there are exactly $\deg_{v_i}(\tilde{f}) - 1 \geq 1$ edges in Δ that connect v_i to v_∞ for $i \neq \infty$, where $\deg_x(\tilde{f})$ denotes the local degree of \tilde{f} at $x \in \Gamma'$.*
- (5) *$\sum_{x \in \Gamma'} (\deg_x(\tilde{f}) - 1) \leq 2d_\Gamma - 2$.*
- (6) *The graph $\overline{\Gamma \setminus \Delta}$ is connected.*

Though the extension \tilde{f} is essential in the definition above, we will not reference it again. This is because our combinatorial invariant includes additional edges (from Hubbard trees and Newton rays), and we wish to extend the graph map for this more refined invariant.

Next we define abstract Newton rays. Let Γ be a finite connected graph embedded in \mathbb{S}^2 and $f : \Gamma \rightarrow \Gamma$ a weak graph map so that after f is promoted to a graph map in the sense of Remark 3.6, it can be extended to a branched cover $\tilde{f} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$.

Definition 4.3. A periodic abstract Newton ray R with respect to (Γ, f) is an arc in \mathbb{S}^2 that satisfies the following:

- *one endpoint of R is a vertex $i(R) \in \Gamma$ (the “initial endpoint”), and the other endpoint is a vertex $t(R) \in \mathbb{S}^2 \setminus \Gamma$ (the “terminal endpoint”).*
- *$R \cap \Gamma = \{i(R)\}$.*
- *there is a minimal positive integer m so that $\tilde{f}^m(R) = R \cup \mathcal{E}$, where \mathcal{E} is a finite union of edges from Γ .*
- *for every $x \in R \setminus t(R)$, there exists an integer $k \geq 0$ such that $\tilde{f}^k(x) \in \Gamma$.*

We say that the integer m is the period of R , and that R lands at $t(R)$.

Definition 4.4. A preperiodic abstract Newton ray R' with respect to (Γ, f) is an arc in \mathbb{S}^2 which satisfies the following:

- *one endpoint of R' is a vertex $i(R') \in \Gamma$, and the other endpoint is $t(R') \in \mathbb{S}^2 \setminus \Gamma$.*
- *$R' \cap \Gamma = \{i(R')\}$.*
- *there is a minimal integer $l > 0$ such that $\tilde{f}^l(R') = R \cup \mathcal{E}$, where \mathcal{E} is a finite union of edges of Γ and R is a periodic abstract Newton ray with respect to (Γ, f) .*
- *R' is not a periodic abstract Newton ray with respect to (Γ, f) .*

We say that the integer l is the preperiod of R' , and that R' lands at $t(R')$.

Now we are ready to introduce the concept of an abstract extended Newton graph. Later we prove that this graph carries enough information to characterize postcritically finite Newton maps.

Definition 4.5 (Abstract extended Newton graph). *Let $\Sigma \subset \mathbb{S}^2$ be a finite connected graph, and let $f : \Sigma \rightarrow \Sigma$ be a weak graph map. A pair (Σ, f) is called an abstract extended Newton graph if the following are satisfied:*

- (1) (Edge Types) *Any two different edges in Σ may only intersect at vertices of Σ . Every edge must be one of the following three types (defined respectively in items (2), (3-4), and (6-7) below):*
 - *Type N: An edge in the abstract Newton graph Γ*
 - *Type H: An edge in a periodic or pre-periodic abstract Hubbard tree*
 - *Type R: A periodic or pre-periodic abstract Newton ray with respect to (Γ, f) .*
- (2) (Abstract Newton graph) *There exists a positive integer N and an abstract Newton graph Γ at level N so that $\Gamma \subseteq \Sigma$. Furthermore N is minimal so that condition (5) holds.*
- (3) (Periodic Hubbard trees) *There is a finite collection of (possibly degenerate) minimal abstract extended Hubbard trees $H_i \subset \Sigma$ which are disjoint from Γ , and for each H_i there is a minimal positive integer $m_i \geq 2$ called the period of the tree such that $f^{m_i}(H_i) = H_i$.*
- (4) (Preperiodic trees) *There is a finite collection of possibly degenerate trees $H'_i \subset \Sigma$ of preperiod ℓ_i , i.e. there is a minimal positive integer ℓ_i so that $f^{\ell_i}(H'_i)$ is a periodic Hubbard tree (H'_i is not necessarily a Hubbard tree). Furthermore for each i , the tree H'_i contains a critical or postcritical point.*
- (5) (Trees separated) *Any two different periodic or pre-periodic Hubbard trees lie in different complementary components of Γ .*
- (6) (Periodic Newton rays) *For every periodic abstract extended Hubbard tree H_i of period m_i , there is exactly one periodic abstract Newton ray R_i connecting H_i to Γ . The ray lands at a repelling fixed point $\omega_i \in H_i$ and has period $m_i \cdot r_i$ where r_i is the period of any external ray landing at the corresponding fixed point of the polynomial realizing H_i .*
- (7) (Preperiodic Newton rays) *For every preperiodic tree H'_i , there exists at least one preperiodic abstract Newton ray in Σ connecting H'_i to Γ . A preperiodic ray landing at a periodic Hubbard tree must have preperiod 1.*
- (8) (Unique extendability) *After promoting the weak graph map f to a graph map $\hat{f} : \hat{\Sigma} \rightarrow \Sigma$ as in Remark 3.6, the conditions of Proposition 3.4 are met; thus \hat{f} has a regular extension \bar{f} which is unique up to Thurston equivalence.*
- (9) (Topological admissibility) *The total number of critical points of f in Σ counted with multiplicity is $2d_\Gamma - 2$, where d_Γ is the degree of the abstract channel diagram $\Delta \subset \Gamma$.*

Remark 4.6 (Vertices and mapping properties of the graph). The set of vertices of the extended Newton graph is taken to be the union of all Hubbard

tree and Newton graph vertices. It is a consequence of the axioms that all critical, postcritical, and fixed points of the extension \bar{f} are vertices. Also implied is the fact that \bar{f} maps Hubbard tree edges only to Hubbard tree edges, and any point of Σ not contained in a Hubbard tree is eventually mapped into the channel diagram Δ .

Remark 4.7 (Implied auxiliary edges). Suppose that H_i is a Hubbard tree (or Hubbard tree preimage) in some complementary component U_i of Γ with connecting Newton ray R_i . If H_i contains a critical point, the existence of a regular graph map extension from Condition (8) implies that Σ must have at least one pre-periodic Newton ray edge distinct from R_i connecting H_i to Γ . All such pre-periodic rays must map to $f(R_i)$ under f (ignoring the parts in Γ as usual), and each such ray is called an *auxiliary edge corresponding to R_i* .

5. EQUIVALENCE OF ABSTRACT EXTENDED NEWTON GRAPHS

When the extended Newton graph was constructed for a postcritically finite Newton map in [LMS], the Newton graph and Hubbard tree edges were constructed intrinsically, but the construction of the Newton rays involved a choice. Both endpoints of the Newton rays were chosen arbitrarily, and even after the endpoints have been fixed, there are a countably infinite number of homotopy classes of arcs by which a non-degenerate Hubbard tree could be connected to the Newton graph (corresponding to the fact that removing the Hubbard tree from the complementary component of the Newton graph produces a topological annulus).

Let (Σ_1, f_1) and (Σ_2, f_2) be two abstract extended Newton graphs. In this section, we define an equivalence relation for abstract extended Newton graphs so that we can tell from the combinatorics of (Σ_1, f_1) and (Σ_2, f_2) whether or not their extensions to branched covers are Thurston equivalent.

For an abstract Newton graph Σ , denote by Σ^- the resulting graph when all edges of type R are removed; only the type N and H edges remain. We keep the endpoints of the removed edges as vertices of Σ^- . The purpose of removing the type R edges is that their construction involved choices, while Σ^- is the part constructed without choices.

Remark 5.1 (Some simplifying assumptions). To simplify notation, we assume for the rest of Section 5 that Σ_1 and Σ_2 are combinatorially and dynamically equal apart from their Newton rays. Specifically this means that the identity map on \mathbb{S}^2 induces a graph isomorphism between the Newton graphs of Σ_1 and Σ_2 (from now on denoted Γ), as well as the Hubbard trees. We also assume that $f_1 = f_2$ on all vertices of Σ_1 and Σ_2 (the restriction to vertices of either graph map will be denoted f).

The combinatorial equivalence given in Definition 5.7 must somehow encode the Thurston class of graph map extensions to complementary components of Σ_1^- and Σ_2^- that contain non-degenerate Hubbard trees (for other types of components it is already clear how to proceed). This is our primary focus from now until the definition is given. The fundamental object of study are grand orbits of Newton rays.

Definition 5.2 (Newton ray grand orbit). *Let R be a Newton ray. The (forward) orbit of R is the set of all Newton rays R' so that for some k , we have $\bar{f}^k(R) = R' \cup \mathcal{E}$ where \mathcal{E} is a union of edges from Γ . The grand orbit of a Newton ray edge R in Σ is the union of all Newton ray edges whose orbits intersect the orbit of R .*

In Section 5.1 we describe how to alter the endpoints and accesses of Newton ray grand orbits in Σ_1 so that they coincide with those of Σ_2 (without changing the homotopy class of the graph map extension). Once this is done, a method is given in Section 5.2 to determine whether the rays determine equivalent extensions across complementary components of Σ_1^- and Σ_2^- that contain non-degenerate Hubbard trees; accordingly, an equivalence is placed on Newton ray grand orbits. Finally the combinatorial equivalence of abstract extended Newton graphs is given in Section 5.3 in terms of the equivalence on Newton ray grand orbits.

5.1. Making Newton ray endpoints and acceses coincide. Here we perform two initial alterations to the Newton grand orbits of Σ_1 and Σ_2 so that their endpoints and accesses to Hubbard tree vertices coincide (it is indeed possible for a repelling fixed point of a polynomial to be the landing point of multiple external rays, and we simply wish to fix a preferred external ray, corresponding to an access to the fixed point in the complement of the filled Julia set). These alterations are done so as to not change the homotopy classes of the graph map extensions (see Figure 4).

Let H_r be a periodic Hubbard tree in both Σ_1 and Σ_2 which is contained in the complementary component U_r of Γ . Let $R_{1,r} \subset \Sigma_1$ be a Newton ray landing at $\omega_1 \in H_r$, and let $R_{2,r} \subset \Sigma_2$ be a ray landing at $\omega_2 \in H_r$.

Step 1) *Without changing the homotopy class of the extension to \mathbb{S}^2 , use the following method to replace the ray grand orbit of $R_{2,r}$ by a ray grand orbit whose **landing points on Hubbard trees** and accesses coincide with those of the grand orbit of $R_{1,r}$.* Let $H_{r+1} = f(H_r) \subset V_r$, where V_r is the complementary component of $f(\Gamma)$ containing H_{r+1} . Connect $f(\omega_1)$ to some boundary vertex of V_r by an arc $R'_{2,r+1}$ in $V_r \setminus f_2(R_{2,r})$. Let the new Newton ray $R'_{2,r}$ be an arc that connects ω_1 through the desired access to a vertex in ∂U_r which maps under f to the endpoint of $R'_{2,r+1}$ in ∂V_r . The interior of $R'_{2,r}$ is not permitted to intersect H_r or the auxiliary rays corresponding to $R_{2,r}$ (see Remark 4.7). Using an identical procedure, this new choice of connecting ray can be propagated inductively through the grand orbit of H_r in Σ by lifting. For convenience we now relabel $R'_{2,r}$ as $R_{2,r}$.

Step 2) *Without changing the homotopy class of the extension to \mathbb{S}^2 , use the following method to replace the grand orbit of the ray $R_{2,r}$ by a ray grand orbit, both of whose **endpoints** and accesses are all identically those of the grand orbit of $R_{1,r}$.* Let α_{r+1} be the arc produced by traversing ∂V_r in the clockwise direction starting at $f_2(R_{2,r}) \cap \partial V_r$ until the endpoint of $f_1(R_{1,r})$ in ∂V_r is reached for the first time. There are two ways (up to homotopy) to connect the end vertex of α_{r+1} to $f(\omega_1)$ in $V_r \setminus f_2(R_{2,r})$. Let the new Newton ray $R'_{2,r+1}$ be the choice of this arc so that the sector $S_{f_2(R_{2,r})}$ in V_r

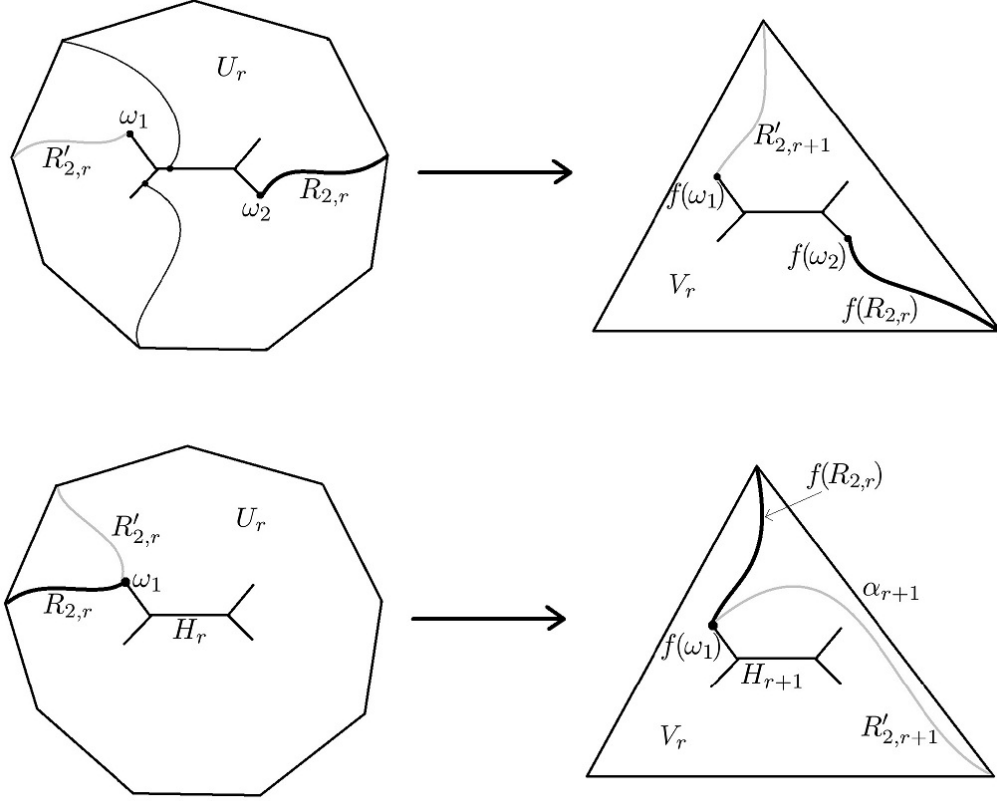


FIGURE 4. The top half of this figure illustrates Step 1, and the lower half illustrates Step 2 for a degree three Hubbard tree. Auxiliary rays corresponding to $R_{2,r}$ in Step 1 are indicated by thinner solid black curves.

bounded by α_{r+1} , $f_2(R_{2,r})$, and $R'_{2,r+1}$ satisfies the following:

$$\overline{S_{f_2(R_{2,r})}} \cap H_{r+1} = f(\omega_1).$$

In a corresponding way under pullback, we produce an arc $R'_{2,r} \subset U_r$. Propagate these choices through the grand orbit of $R_{2,r}$ in Σ by lifting. Note that when lifting to a strictly preperiodic tree, there may be multiple choices of the lift depending on the ray. Choose the lift that has the same endpoints as some arc in $R_{1,r}$. Again for convenience, relabel $R'_{2,r}$ as $R_{2,r}$.

As indicated earlier, what we have achieved so far is to change the Newton rays so that the homotopy class of the extension is unchanged. The Newton rays are now “comparable” in a sense that we exploit in the next section.

5.2. Equivalence on Newton ray grand orbits. We now wish to compare the extensions of graph maps over complementary components of Σ_1^- and Σ_2^- containing nondegenerate Hubbard trees in their closure. The other complementary components may only be disks or once-punctured disks; these are not discussed here because there is a unique extension over such components up to isotopy given by the Alexander trick [FM11, Chapter 2.2].

Restricting attention to the complementary components of Γ that contain the grand orbit of a single Hubbard tree, we observe that whether or not two extensions are equivalent (in the sense of Definition 5.3) can be determined solely in terms of the Newton rays. We thus define a combinatorial equivalence on Newton ray grand orbits so that two ray grand orbits are Thurston equivalent if and only if the extensions to the complementary components of Γ intersecting the grand orbit are equivalent.

Let H_1 be a nondegenerate Hubbard tree in Σ_1 (and Σ_2) of preperiod $r \geq 0$ and period $m \geq 2$, and let $H_i = f^{i-1}(H_1)$ for $1 \leq i \leq r+m$. Let U_i be the complementary component of Σ_1^- that contains H_i and let $\mathcal{U} = \cup_i U_i$.

Definition 5.3 (Thurston equivalent graph extensions over \mathcal{U}). *We say that two extensions $\overline{f_1}, \overline{f_2}$ to \mathcal{U} of the graph maps f_1, f_2 are Thurston equivalent over \mathcal{U} if there are homeomorphisms $\phi_1, \phi_2 : \overline{\mathcal{U}} \rightarrow \overline{\mathcal{U}}$ so that:*

$$\phi_1 \circ \overline{f_1} = \overline{f_2} \circ \phi_2$$

and there exists a homotopy $\Phi : [0, 1] \times \overline{\mathcal{U}} \rightarrow \overline{\mathcal{U}}$ with $\Phi(0, \cdot) = \phi_1$, $\Phi(1, \cdot) = \phi_2$ so that for all $t \in [0, 1]$, we have that $\Phi(t, \cdot)|_{\partial \mathcal{U}} \subset \partial \mathcal{U}$ and $\Phi(t, \cdot)$ restricts to the identity on graph vertices for all t . If ϕ_1 and ϕ_2 are homotopic to the identity in the sense just mentioned, the extensions are said to be homotopic over \mathcal{U} .

Each Hubbard tree H_i is contained in a complementary component of Γ which is a topological annulus when H_i is removed. Let T_i denote the right-hand Dehn twist about this annulus. The core arcs of all such T_i are pairwise disjoint because no two H_i lie in the same complementary component of the Newton graph. Thus any two such twists commute.

Let $R_{1,i}, R_{2,i}$ for $1 \leq i \leq r+m$ denote the Newton ray edges connecting H_i to Γ in Σ_1, Σ_2 respectively. Recall that after applying the two steps of Section 5.1, we may assume that $R_{1,i}$ and $R_{2,i}$ have the same endpoints. The equality symbol is used for arcs to indicate that they are homotopic in U_i rel endpoints. Note that for all i , there is a unique $\ell_i \in \mathbb{Z}$ and $\ell'_i \in \mathbb{Z}$ so that $T_i^{\ell_i}(R_{1,i}) = R_{2,i}$ and $T_{i+1}^{\ell'_i}(\overline{f_1}(R_{1,i})) = \overline{f_2}(R_{2,i})$.

Lemma 5.4 (Numerics of equivalent extensions). *The extensions $\overline{f_1}$ and $\overline{f_2}$ over \mathcal{U} of the graph maps f_1 and f_2 are Thurston equivalent if and only if there are integers n_1, \dots, n_{r+m-1} that satisfy the following system of linear equations:*

$$(1) \quad d_i(n_i - \ell_i) + \ell'_i = n_{i+1}$$

where $1 \leq i \leq r+m-1$ and $n_r = n_{r+m}$.

Proof. Suppose that the extensions $\overline{f_1}$ and $\overline{f_2}$ are equivalent. Then there are $n_i \in \mathbb{Z}$ so that up to branched cover homotopy,

$$(2) \quad S \circ \overline{f_1} = \overline{f_2} \circ S$$

where $S = T_1^{n_1} \circ \dots \circ T_{m+r-1}^{n_{m+r-1}}$.

Fix i as in the statement of the lemma. All of the Dehn twists T_1, \dots, T_{m+r-1} fix $\overline{f_1}(R_{1,i})$ except possibly T_{i+1} , and thus the expression on the left side of Equation 2 acts on the ray $R_{1,i}$ as follows:

$$(3) \quad S \circ \overline{f_1}(R_{1,i}) = T_{i+1}^{n_{i+1}} \circ \overline{f_1}(R_{1,i})$$

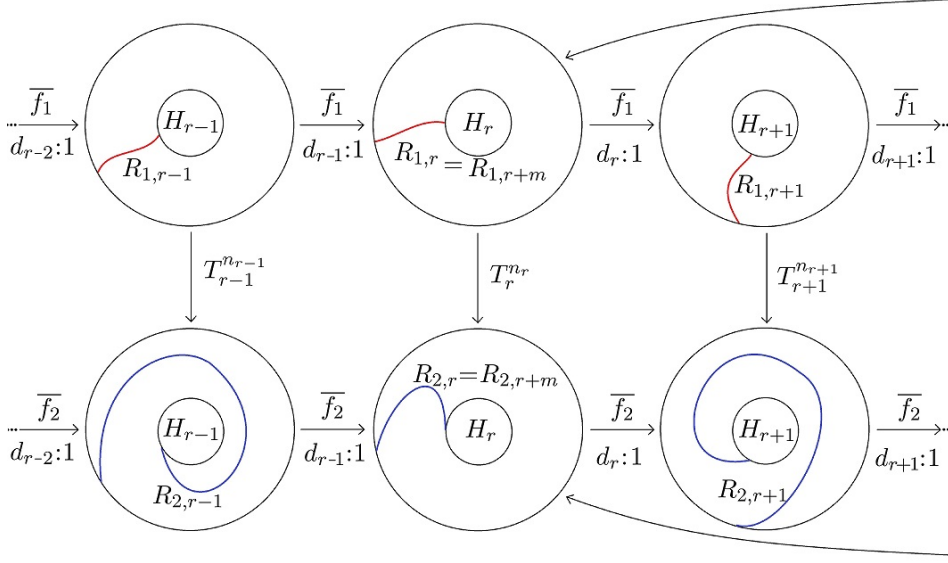


FIGURE 5. Topological picture of the annuli over which f is extended in Lemma 5.4

and the right side acts on $R_{1,i}$ as follows:

$$\begin{aligned}
 \bar{f}_2 \circ S(R_{1,i}) &= \bar{f}_2(T_i^{n_i}(R_{1,i})) \\
 &= \bar{f}_2(T_i^{n_i-\ell_i}(R_{2,i})) \\
 &= T_{i+1}^{d_i(n_i-\ell_i)}(\bar{f}_2(R_{2,i})) \\
 &= T_{i+1}^{d_i(n_i-\ell_i)+\ell'_i}(\bar{f}_1(R_{1,i})).
 \end{aligned}$$

Equating the expression in the previous line and the right side of Equation 3 we obtain Equation 1.

If on the other hand Equation 1 has integer solutions, it follows that $S \circ \bar{f}_1$ and $\bar{f}_2 \circ S$ are homotopic over \mathcal{U} using a close analog of Lemma 3.3. Thus the extensions \bar{f}_1 and \bar{f}_2 are Thurston equivalent over \mathcal{U} . \square

Remark 5.5. Note that Definition 5.3 can easily be extended to the case of complementary components of the Newton graph containing the grand orbit of a Hubbard tree in the extended Newton graph instead of just the orbit. Similar numerics as in the previous lemma hold for this slightly more general case.

Definition 5.6 (Newton ray grand orbit equivalence). *We say that the grand orbit of the Newton ray $R_{1,r} \subset \Sigma_1$ landing at H_r is equivalent to the grand orbit of the ray $R_{2,r} \subset \Sigma_2$ landing at H_r if the numerical condition of Lemma 5.4 is satisfied.*

5.3. Equivalence on abstract extended Newton graphs. We now define the combinatorial equivalence relation on abstract extended Newton graphs that is used in the classification theorem (Theorem 1.5) and prove an important result connecting this equivalence with Thurston equivalence.

Definition 5.7 (Equivalence relation for abstract extended Newton graphs). *Let (Σ_1, f_1) and (Σ_2, f_2) be two abstract extended Newton graphs with self-maps $f_i: \Sigma_i \rightarrow \Sigma_i$, for $i = 1, 2$. We say that (Σ_1, f_1) and (Σ_2, f_2) are equivalent if*

- *there exist two homeomorphisms $\phi_1, \phi_2: \Sigma_1^- \rightarrow \Sigma_2^-$ that preserve the cyclic order of edges at all the vertices of Σ_1^-, Σ_2^-*
- *the equation $\phi_1 \circ f_1 = f_2 \circ \phi_2$ holds on Σ_1^-*
- *ϕ_1 is isotopic to ϕ_2 relative to the vertices of Σ_1*
- *all Newton ray grand orbits are equivalent.*

It will now be shown that two abstract extended Newton graphs are combinatorially equivalent if and only if their extensions are Thurston equivalent.

Lemma 5.8 (Combinatorial formulation of Thurston equivalence). *Let (Σ_1, f_1) and (Σ_2, f_2) be abstract extended Newton graphs with graph map extensions $\bar{f}_1, \bar{f}_2: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ respectively. Then (Σ_1, f_1) and (Σ_2, f_2) are equivalent if and only if (\bar{f}_1, Σ_1') and (\bar{f}_2, Σ_2') are equivalent as marked branched covers.*

Proof. First assume that the two abstract extended Newton graphs are equivalent. Let Σ_1^-, Σ_2^- denote the corresponding graphs with Newton ray edges removed, and observe that the complementary components of Σ_1^-, Σ_2^- are homeomorphic either to disks or annuli.

By definition of extended Newton graph equivalence, there are graph homeomorphisms $\phi_1, \phi_2: \Sigma_1^- \rightarrow \Sigma_2^-$ such that

$$\phi_1(f_1(z)) = f_2(\phi_2(z))$$

for all $z \in \Sigma_1^-$ and ϕ_1, ϕ_2 preserve the cyclic order of edges at all the vertices of Σ_1^- .

Using Steps 1 and 2 of Section 5.1, we may assume all Newton rays of Σ_1 and Σ_2 have corresponding endpoints (without changing the homotopy class of the resulting extension). Then since the Newton ray grand orbits are equivalent in the sense of Definition 5.6, there must be some S_1 and S_2 which are both products of Dehn twists about the non-degenerate Hubbard trees as in Lemma 5.4 so that

$$(4) \quad S_1 \circ \phi_1 \circ f_1(z) = f_2 \circ \phi_2 \circ S_2(z)$$

for all $z \in \Sigma_1$ and S_1 is homotopic to S_2 relative to the vertices of Σ_1 . The maps on both sides of Equation 4 have regular extensions (see Proposition 3.4) and they also satisfy the hypotheses of Lemma 3.3. Thus f_1 and f_2 are equivalent as marked branched covers

Now suppose that (\bar{f}_1, Σ_1') and (\bar{f}_2, Σ_2') are equivalent as marked branched covers. Take $g_0, g_1: (\mathbb{S}^2, \Sigma_1') \rightarrow (\mathbb{S}^2, \Sigma_2')$ to be the maps from the definition of Thurston equivalence where $g_0 \circ \bar{f}_1 = \bar{f}_2 \circ g_1$. Let e be an edge of Σ_1^- with endpoints ∂e . Then $g_1(e)$ connects the two points in $g_1(\partial e)$. Moreover, g_1 preserves the cyclic order at each vertex of Σ_1^- , because it is an orientation preserving homeomorphism of \mathbb{S}^2 . Let $g': (\mathbb{S}^2, \Sigma_2') \rightarrow (\mathbb{S}^2, \Sigma_2')$ be a homeomorphism that for all edges e not Newton rays maps each $g_1(e)$ to an edge of Σ_2 that connects the two points in $g_1(\partial e)$.

Then $g' \circ g_1$ realizes an equivalence between the two abstract extended Newton graphs (let $\phi_0 = \phi_1 = g' \circ g_1$ in Definition 5.7), except that the Newton rays must still be shown to be equivalent. Apply Steps 1 and 2 in Definition 5.6 so that all Newton rays have corresponding endpoints under ϕ_0, ϕ_1 . Then since $\overline{f_1}$ and $\overline{f_2}$ are Thurston equivalent as branched covers, Lemma 5.4 implies the rays are equivalent. Thus (Σ_1, f_1) and (Σ_2, f_2) are combinatorially equivalent. \square

6. NEWTON MAPS FROM ABSTRACT EXTENDED NEWTON GRAPHS

We now prove that all abstract extended Newton graphs are realized by Newton maps.

Proof of Theorem 1.4. It suffices to show that the marked branched cover (\overline{f}, Σ') is unobstructed, where Σ' denotes the set of vertices of Σ . The conclusion will follow by Head's theorem, where the holomorphic fixed point theorem is used to argue that the point at infinity is repelling [Mil06].

Suppose to the contrary that Π is a Thurston obstruction for (\overline{f}, Σ') , and without loss of generality assume Π is irreducible. Recall from Condition (2) that Σ contains an abstract Newton graph Γ which in turn contains an abstract channel diagram Δ . The following lemma restricts where obstructions may exist, using Theorem 2.5.

Lemma 6.1. *If Π is a Thurston obstruction for (\overline{f}, Σ') , then*

$$\Pi \cdot (\Gamma \setminus \Delta) = 0.$$

Furthermore, if R_i is a type R edge, $\Pi \cdot R_i = 0$.

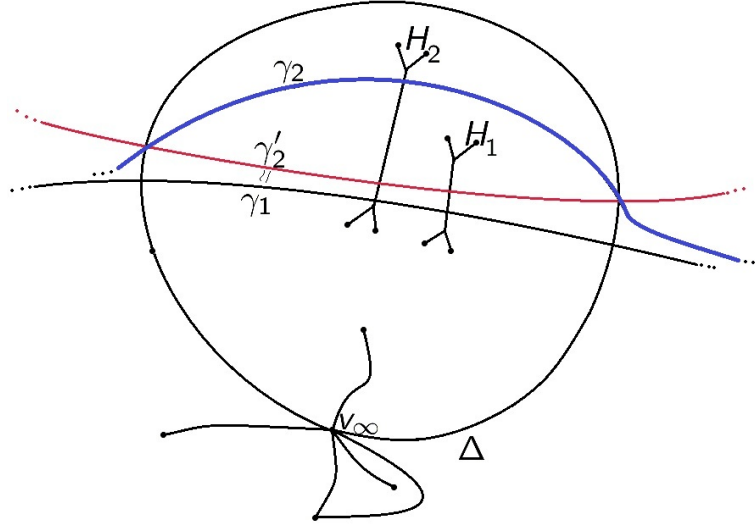
Proof. Let λ be an edge in Δ so that $\lambda \cdot \Pi \neq 0$. Since $\{\lambda\}$ itself forms an irreducible arc system, the second case of Theorem 2.5 implies that Π intersects no other preimage of λ except for λ itself. If on the other hand, $\lambda \cdot \Pi = 0$, the first case of Theorem 2.5 implies that no preimage of λ intersects Π . Since every edge in Γ is a preimage of an edge in Δ , and every type R edge is composed of preimages of Δ , the conclusion follows. \square

The proof of the theorem is now completed by showing that whether or not Π has intersection with Δ in minimal position, a contradiction results.

Contradiction for the case $\Pi \cdot \Delta \neq 0$

Let γ_1 be any curve in Π so that $\gamma_1 \cdot \Delta \neq 0$. Recall from Definition 4.2 that $\overline{\Gamma \setminus \Delta}$ is connected and may not intersect Π ; thus $\overline{\Gamma \setminus \Delta}$ must be a subset of one of the complementary components of γ_1 . Denote the complementary component of γ_1 that does not contain $\overline{\Gamma \setminus \Delta}$ by $D(\gamma_1)$. None of the vertices of Γ except possibly v_∞ lie in $D(\gamma_1)$. However, there must be at least two vertices of Σ in $D(\gamma_1)$ for otherwise γ_1 would not be essential. The only vertices which could possibly be in $D(\gamma_1)$ are v_∞ and Hubbard tree vertices. No Hubbard trees may be a subset of $D(\gamma_1)$, because the Hubbard tree must be connected to a vertex in $\Gamma \setminus \Delta$ by a Newton ray, and Newton rays may not intersect γ_1 . But since there must be at least one Hubbard tree vertex in $D(\gamma_1)$, there must be some Hubbard tree H_1 so that $H_1 \cdot \gamma_1 \neq 0$.

Let $\gamma_2 \in \Pi$ be some curve whose preimage under \overline{f} has a component γ'_2 which is homotopic to γ_1 rel vertices (γ_2 exists by irreducibility). Clearly γ'_2

FIGURE 6. Illustration of the first case $\Pi \cdot \Delta \neq 0$

must intersect H_1 and Δ . Let λ_1 be some component of $\gamma'_2 \cap (S^2 \setminus \Delta)$ that intersects H_1 . Recall that H_1 must have period at least two, so $H_2 := \bar{f}(H_1)$ may not intersect H_1 . Also H_1 and H_2 must be in the same complementary component of Δ because λ_1 connects H_1 to Δ (without passing through any other edges of Γ due to Lemma 6.1), and \bar{f} is an orientation preserving map that fixes each edge of Δ .

Now we show that H_1 and H_2 can be connected by some path that avoids Γ except at its endpoints. Since \bar{f} fixes the edges of Δ , it must be that $\lambda_2 := \bar{f}(\lambda_1)$ has the same endpoints as λ_1 . Note also that λ_2 may intersect Δ only at these points since λ_1 may not intersect any preimages of Δ (an immediate consequence of the proof of Lemma 6.1). Starting at the closest intersection of H_1 and λ_1 to one of the endpoints of λ_1 , traverse λ_1 until right before the intersection with the edge of Δ . Traverse a path in a small neighborhood of this endpoint that leads to λ_2 without intersecting any edges of Γ . Traverse λ_2 until H_2 is reached. This completes the construction of a path from H_1 to H_2 avoiding Γ , which contradicts the assumption that H_1 and H_2 were separated by the Newton graph (Condition (5)).

Contradiction for the case $\Pi \cdot \Delta = 0$

Using Lemma 6.1 we see that $\Pi \cdot \Gamma = 0$. Recall the assumption that every complementary component of Γ contains at most one abstract extended Hubbard tree (Condition (5)).

Suppose that U is such a complementary component containing some $\gamma \in \Pi$. The only postcritical points that could possibly be contained in U are vertices of Hubbard trees, so U contains one Hubbard tree or one Hubbard tree preimage. Since γ is periodic, the Hubbard tree must in fact be periodic. Thus U contains exactly one periodic abstract Hubbard tree H of some period m .

Define $F := \bar{f}^m$, and note that Π is also a Thurston obstruction for F . We will show that the two Thurston linear maps F_Π and $(F|_U)_\Pi$ are equal,

contradicting the realizability of abstract Hubbard trees [Poi93, Theorem II.4.7].

We can extract an irreducible Thurston obstruction for F from Π , which we again denote by Π , and assume that U still contains some component of Π . To show that $F_\Pi = (F|_U)_\Pi$, we show that in fact, $\Pi \subset U$. Suppose that W is a complementary component of Γ different from U , and $\gamma' \subset W$ for some $\gamma' \in \Pi$. By the irreducibility of Π , there is some $n > 0$ and a component γ'' of $F^{-n}(\gamma')$ that is homotopic to γ rel vertices. Note that $\gamma'' \subset U$ and that its complementary component that is a subset of U contains some vertices of Σ which must in fact be vertices of H .

There are two cases to consider depending on whether γ'' intersects H or not. If $\gamma'' \cdot H = 0$, then H is a subset of the complementary component of γ'' that is contained in U . But H must be connected to Γ by an edge of type R (Condition (6) and (7)) which must then intersect γ . This is impossible since Theorem 2.5 implies that type R edges cannot intersect γ . On the other hand, if $\gamma'' \cdot H \neq 0$, then $F(\gamma'') \cap F^n(H) \neq \emptyset$. Since $F^n(H) = H$, we obtain that $\gamma' \cap H \neq \emptyset$. This contradicts the fact that $U \neq W$ since $\gamma' \subset W$ and $H \subset U$. \square

7. PROOF OF THE CLASSIFICATION THEOREM

Theorem 1.2 of [LMS] asserts that every postcritically finite Newton map has an extended Newton graph that satisfies the axioms of Definition 4.5, and we have shown in Section 6 that every abstract extended Newton graph extends to an unobstructed branched cover, and is therefore realized by a Newton map. We now check that these two assignments are well-defined on equivalence relations and are inverses of each other, giving an explicit classification of postcritically finite Newton maps in terms of combinatorics.

Denote by \mathcal{N} the set of postcritically finite Newton maps up to affine conjugacy. Denote by \mathcal{G} the set of abstract extended Newton graphs up to Thurston equivalence (Definition 5.7). Equivalence classes in both cases are denoted by square brackets. Our first goal is to show that the assignments made in Theorems 1.3 and 1.4 are well-defined on the level of equivalence classes, namely, they induce mappings $\mathcal{F} : \mathcal{N} \rightarrow \mathcal{G}$ and $\mathcal{F}' : \mathcal{G} \rightarrow \mathcal{N}$.

We now argue that \mathcal{F} is well-defined. The construction from [LMS] of the extended Newton graph for a fixed Newton map involved no choices in the construction of type H and N edges, and possibly many choices in the construction of type R edges. Let $(\Delta_{N,1}^*, N_p)$ and $(\Delta_{N,2}^*, N_p)$ be two extended Newton graphs constructed for N_p . Proposition 6.4 in [LMS] asserts that $\Delta_{N,1}^- = \Delta_{N,2}^-$ and $N_p|_{\Delta_{N,1}^-} = N_p|_{\Delta_{N,2}^-}$ (recall that $\Delta_{N,1}^-$ denotes the graph $\Delta_{N,1}$ with all Newton ray edges removed). We thus only need to show that the Newton ray grand orbits are equivalent. The branched cover $(N_p, (\Delta_{N,1}^*)')$ is identical as a branched cover to $(N_p, (\Delta_{N,2}^*)')$ and they are both extensions of graph maps $N_p|_{\Delta_{N,1}^*}$ and $N_p|_{\Delta_{N,2}^*}$ respectively. Lemma 5.8 then implies equivalence for corresponding ray grand orbits.

Well-definedness of \mathcal{F}' is immediate from the fact that equivalent graphs have Thurston equivalent extensions (Lemma 5.8) which correspond to affine conjugate Newton maps by Thurston rigidity (Theorem 2.3).

Proof of Theorem 1.5. We first show injectivity of $\mathcal{F} : \mathcal{N} \rightarrow \mathcal{G}$. Let N_{p_1} and N_{p_2} be two postcritically finite Newton maps that have equivalent extended Newton graphs $\Delta_{N,1}^*$ and $\Delta_{N,2}^*$. Theorem 1.3 asserts that each of these graphs satisfy the axioms of an abstract extended Newton graph, and since both graphs are equivalent, the marked branched covers $(N_{p_1}, (\Delta_{N,1}^*)')$ and $(N_{p_2}, (\Delta_{N,2}^*)')$ are equivalent by Lemma 5.8. We may then conclude that N_{p_1} and N_{p_2} are affine conjugate using Thurston rigidity.

Next we show injectivity of $\mathcal{F}' : \mathcal{G} \rightarrow \mathcal{N}$. Suppose that a postcritically finite Newton map N_p realizes two abstract extended Newton graphs (Σ_1, f_1) and (Σ_2, f_2) . By minimality of the extended Hubbard trees and the Newton graph, we know that $\Sigma_1' = \Sigma_2'$. Then the marked branched covers (N_p, Σ_1') and (N_p, Σ_2') are equivalent. By Lemma 5.8 we conclude that (Σ_1, f_1) and (Σ_2, f_2) are combinatorially equivalent.

Finally we prove that \mathcal{F} and \mathcal{F}' are bijective and inverses of each other. Let $(\Sigma, f) \in \mathcal{G}$ be an abstract extended Newton graph. It follows from Theorem 1.4 that (Σ, f) is realized by a postcritically finite Newton map N_p . Thus

$$\mathcal{F}'([\Sigma, f]) = [N_p].$$

Denote by Δ_N^* an extended Newton graph associated to N_p which is guaranteed by Theorem 1.3 so that

$$\mathcal{F}([N_p]) = [(\Delta_N^*, N_p)].$$

The injectivity statement just proved implies that under the equivalence of Definition 5.7,

$$[(\Sigma, f)] = [(\Delta_N^*, N_p)]$$

Thus $\mathcal{F} \circ \mathcal{F}'$ is the identity, and consequently the mapping $\mathcal{F} : \mathcal{N} \rightarrow \mathcal{G}$ is bijective and $\mathcal{F}' \circ \mathcal{F}$ is the identity. \square

This completes the combinatorial classification of postcritically finite Newton maps.

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